"On Flapping Flight of Aëroplanes." By Maurice F. FitzGerald, Professor of Engineering, Queen's College, Belfast. Communicated by Professor G. F. FitzGerald, F.T.C.D., F.R.S. Received February 15,—Read March 2, 1899.

It has been long known, principally through experiments on soaring, that a large, if not by far the largest, part of the supporting force obtained by birds in regular flight, is probably furnished by upward air pressure on their wings, regarded as planes moving horizontally, with their surfaces slightly inclined to the direction of motion.

Langley's "Experiments on Aëro-dynamics" furnish some numerical data for estimating the power required to sustain an aëroplane of given weight, propelled horizontally by a known force, and he applies these to the determination of the problem whether this could be effected by screw propellers, analogous to those of a ship, actuated by machinery of existing type.

The older mathematical investigation by Navier of the problem of flapping flight, seems to be quite discredited, and, indeed, the results brought out always laboured under the physiological difficulty of demanding an amount of horse power per pound of muscle in birds which surpassed, to an almost incredible extent, that available from the better known muscular tissues of other animals. One horse power is 550 ft.-lbs. per second; a horse can ordinarily exert about two-thirds of this, and if we take him to weigh about 10 cwt., we get about a third of a ft.-lb. per second per pound of horse. A man can work at the rate of about 50 ft.-lbs. per second; this, again, is about a third of a ft.-lb. per second per pound of man; and so on with other animals. But, for the flight of birds, it was made out that from about 30 to 1300 ft.-lbs. per second per pound of bird were necessary, which, even after all allowance for higher temperature of their bodies, and large relative mass of wing muscles, compared with those of the limbs of horses, men, &c., seemed highly improbable.

In the following paper an attempt is made to indicate how both progressive and hovering flight may be effected by aëroplanes, attached to a heavy mass, and flapped after the manner of wings, under conditions sufficiently nearly approaching those of Langley's experiments to justify the inference, from his figures, of numerical results, not indeed presumably exact, but sufficiently indicating the order of magnitude of the quantities involved. The figures given may be taken, commercially speaking, as near enough to the truth to enable us to judge whether we are dealing with pounds or with pence, though not exact enough to adjust the change out of half-a-crown in paying for our power.

Let, then, a heavy mass be supposed to be flying through the air

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horizontally, and be supported by wings, consisting of planes of negligible mass, held nearly edgewise to the direction of motion, and moved vertically up and down by some machinery carried by the heavy mass. Let there be in addition some arrangement of the machinery by which the angle of inclination of the planes or wings to their direction of motion is variable. We shall suppose the velocity of progression high, and variations of the propelling force small enough for changes of forward velocity to be neglected, compared to the average forward velocity. This is justified by Langley's experiments, which show that, for small inclinations of an aëroplane, the direct resistance to forward motion is small compared to the supporting force. We shall also, in the first instance, neglect direct air resistance on the heavy mass as small compared with that on the aëroplanes; and, for convenience, we shall call the heavy mass the bird's body and the attached aëroplane the wings.

The case differs from that of a real bird most notably by the circumstances that the aëroplane in our theoretical case has no mass, whereas, a bird's wings have one of very sensible magnitude compared with that of its body, and the aëroplane is supposed to be moved up and down, relatively to the mass, as a whole, instead of being pivoted to it, as a bird's wings are to its body, besides being of constant area, which wings are not. Consequently, as before remarked, numerical results can only be regarded as indicating the order of magnitude of the quantities calculated, not their exact values.

The inclination of the wings being variable, and their motion being compounded of an up-and-down one with a forward one, it is evident that the supporting force may be periodically variable also, and consequently the bird's body will move in a sinuous or wavy path, on the whole horizontal, and the stroke of the wings will be the relative motion of wings and body. If the horizontal velocity be high, and the amplitude of the wing stroke relatively to the air moderate, and the variations in the supporting force not too great, the path of the bird's body will be only slightly waved, so that fig. 1 may be taken as, in a general sort of way, representing what takes place.

In the same figure A_1B_1, A_2B_2, &c., represent the plane of the wing at different successive positions, seen edgewise, and P_1, P_2, &c., the corresponding resultant air pressures on it, in direction and magnitude. Now it is plainly a matter of arrangement of mechanism what these shall be, as they depend on the angle between the lines A_1B_1, &c., and the tangent to the heavy line, marked "Path of wing" at each instant. Consequently it is worth while inquiring what are the conditions of adjustment to cause a forward force on an average to be applied to the bird and wings, when the wings are moved up and down by an engine forming part of the bird's body, when the supporting force is, on an average, equal to the bird's weight, and the forward

force is, on an average, equal to that required, on an average, to propel the aëroplane at its average inclination. Observe that the force P_n depends for its magnitude, not on the actual inclination of the wing path at all, but on the angle between that path and the plane of the wing, while its horizontal component depends on both angles, so that although, as long as there is any supporting force, there is a resistance to forward motion along the wing path, there may nevertheless be a forward force acting horizontally on the whole machine, wings and body taken together, as in the first position shown, for instance.

The thing to do, then, is to formulate this in symbols, and for the present purpose it will suffice to make some simplifying approximations to facilitate the work. We shall then take it that the inclination of the wing path to the horizontal, and the inclination of the plane of the wing to the wing path, are both small enough to assume that for either of them, as well as their sum, the circular measure, the sine, and the tangent do not sensibly differ, and the cosine is unity, for example, for 20° circular measure $= 0{\cdot}349$, sine $= 0{\cdot}342$, tangent $= 0{\cdot}363$, and cosine $= 0{\cdot}94$; so that, up to this at any rate, we shall not be incurring errors exceeding 5 or 6 per cent. by this assumption.

We shall also assume that for a small angle (α) of inclination of an aëroplane to its path, the resultant air pressure is given by the formula $P_\alpha = 2kV^2 \sin\alpha$ in pounds per square foot, where V is velocity in feet per second, and k a coefficient, which, according to Langley, is about $0{\cdot}0017$, and P_α is directed normally to the plane. This agrees nearly with Du Chemins's formula at small angles, as pointed out by Langley.

Now let the angle α be varied according to the law

$$\alpha = m(1 - \mu\cos pt),$$

m and μ being arbitrary constants, p being $2\pi f$ where f is the frequency or number of flaps per second, t being time in seconds.

Then the normal pressure, P_α, on a 'plane of area A, at forward velocity V, inclined at α to its direction of motion is

$$P_\alpha = 2kV^2Am(1 - \mu\cos pt).$$

For convenience we shall take the bird's weight as 1 lb., so that, in general, P_α will be, for any other weight, normal pressure in lbs. per lb. of bird, and A is wing area in square feet per lb. of bird.

Let, at any moment, Z be the height above some datum level of the bird's centre of gravity, and z that of centre of pressure of wings, and suppose that

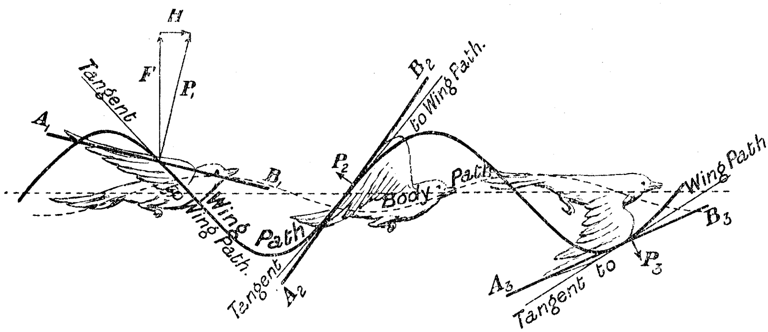
$$z = S\sin(pt + \theta).$$

Then the slope, s, of the wing path is at any moment sensibly

$$s = \frac{pS}{V}\cos(pt + \theta)$$

supposed to be always a small angle.

FIG. 1.



By fig. 1 we see that if F be the vertical supporting force,

$$\mathrm{F} = \mathrm{P}_\alpha \cos(s+\alpha) = \mathrm{P}_\alpha \text{ sensibly},$$

since $(s+\alpha)$ is supposed a small angle; and if H be the horizontal component of P_α,

$$\mathrm{H} = -\mathrm{P}_\alpha \sin(s+\alpha)$$
$$= -\mathrm{P}_\alpha(s+\alpha) \text{ nearly}.$$

This is the resistance to forward motion horizontally, and the work done against it, taking the average rate of working, is

$$\mathrm{U} = \frac{p}{2\pi}\int_0^{2\pi/p} \mathrm{VP}_\alpha(s+\alpha)dt \text{ ft.-lbs. per second.}$$

Again, if the bird's body be rising at the rate $d\mathrm{Z}/dt$, and the wings at a lesser rate, dz/dt, the engine attached to the body, which exerts a downward force, F, on the wings, is doing work, so that, when F is positive, and $\frac{d}{dt}(\mathrm{Z}-z)$ positive, the engine does work, and this is to be the same work, on the average, as that done to overcome the resistance of the air to the plane taken along its actual path, and is

$$\mathrm{W} = \frac{p}{2\pi}\int_0^{2\pi/p} \mathrm{P}_\alpha \frac{d}{dt}(\mathrm{Z}-z)dt.$$

Consequently we get, as a condition for determining some relations among the constants,

$$\int_0^{2\pi/p} \mathrm{VP}_\alpha \alpha dt = \int_0^{2\pi/p} \mathrm{P}_\alpha \frac{d}{dt}(\mathrm{Z}-z)dt \text{ ....(A)}$$

Another condition arises as follows. As F, the vertical force, is sensibly equal to P_α, we have, for the motion of the bird's body,

$$\frac{d^2Z}{dt^2} = g(P_a - 1), \text{ the bird being 1 lb. weight,}$$

$$= g\{2kAV^2m(1-\mu\cos pt) - 1\}$$

and, on integration, all terms containing t^2 or t as factors must vanish, in order for the motion to be, on an average, horizontal. That containing t^2 as a factor will not vanish unless

$$2kAV^2m = 1 \text{ which fixes } m, \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

and gives $\frac{dZ}{dt} = -\frac{\mu g}{p}\sin pt$ as there is no term with t as factor in Z.

By differentiation $\quad \frac{dz}{dt} = pS\cos(pt+\theta),$

whence $\quad \frac{d}{dt}(Z-z) = -\left\{\frac{\mu g}{p}\sin pt + pS\cos(pt+\theta)\right\}.$

Inserting their values in (A) for the quantities involved, we get

$$\int_0^{2\pi/p} \frac{V}{2kAV^2}(1-\mu\cos pt)^2 dt$$

$$= -\int_0^{2\pi/p}(1-\mu\cos pt)\left[\frac{\mu g}{p}\sin pt + pS\cos(pt+\theta)\right]dt.$$

On integration, the only terms that do not vanish give the equation

$$\frac{1}{2kAV} + \tfrac{1}{2}\cdot\frac{\mu^2}{2kAV} = \tfrac{1}{2}\mu pS\cos\theta,$$

which is a quadratic for μ, namely,

$$\mu^2 - 2kAVpS\cos\theta\mu + 2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}$$

Taking the integral of one side alone we find

$$W = \frac{\mu pS\cos\theta}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(D)}$$

Eliminating μ from these we get

$$pS\cos\theta = \pm\frac{W}{\sqrt{(kAVW-\frac{1}{2})}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(E)}$$

If now we draw a curve whose ordinate, y, is $pS\cos\theta$, and abscissa, x, is W, we find it to be of the form shown as AB or CD in fig. 1, with a symmetrical branch below the axis of x, not shown in the figure. On account of the steepness of the first part of the curve CD, that

part of it which extends from W = 2·5 to W = 3 is shown with the scale of abscissæ magnified fifty times, as AB, and corresponding curves for μ are given. The case taken is when AV = 60, so that it applies to flight at 60 feet per second, if the wing area is 1 square foot per pound carried, 30 feet per second with an allowance of 2 square feet per pound, and so on.

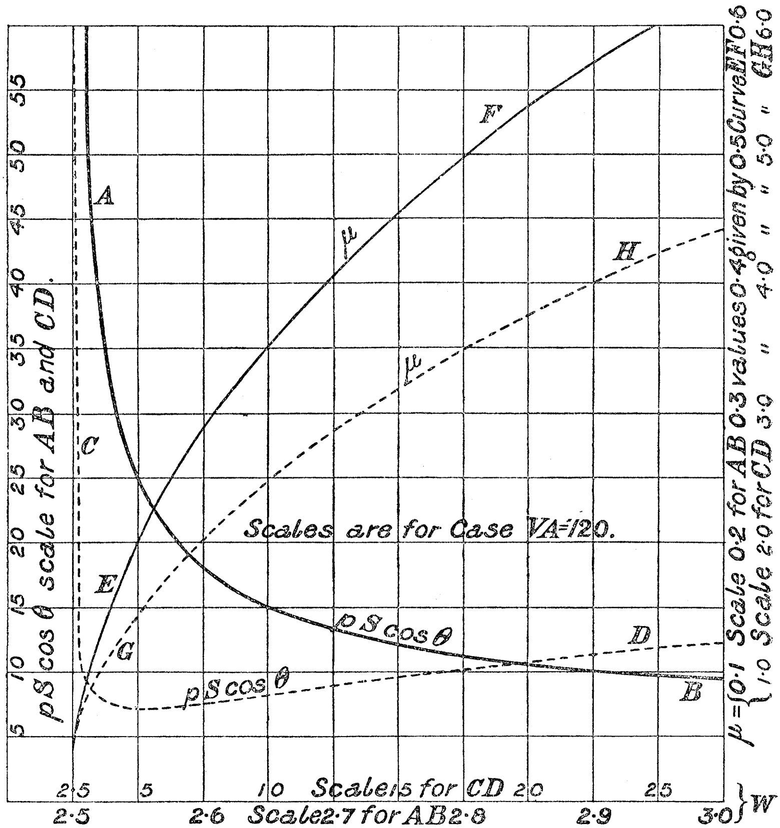
The principal things to note about the curves are that, in the first place, W has a minimum value, in this case 2·5 ft.-lbs. per second per pound carried. This occurs when $p\mathrm{S}\cos\theta$ is infinite, and practically implies that when the rate of flapping is very high, keeping the same stroke, the horse-power required comes very near the minimum, and, remarkably enough, the wing becomes, virtually, exactly equivalent to a plane of area A, inclined at a constant angle $\alpha = m$, moving horizontally, as in Langley's experiments. This is not accidental, but seems to follow from the circumstance that $2\mathrm{W} = \mu p\mathrm{S}\cos\theta$, and if $p\mathrm{S}\cos\theta$ be infinite and W finite, μ must vanish, and, consequently, $\alpha = m(1 - \mu\cos\theta)$ becomes $\alpha = m$ simply.

[*Note added March* 21, 1899.—In the paper as read an erroneous statement here followed, to the effect that the minimum value of W was half that got in Langley's experiments with a plane at angle $\alpha = m$. It arose from the left-hand side in equation (A) being, by mistake, written $\int_0^{2\pi/p}\mathrm{VP}_\alpha(s+\alpha)dt$, instead of as given above. The author is indebted to Prof. J. Purser for pointing out the mistake. The least value of W is equal to that of Langley's experiments.]

Of course, as before mentioned, it is not to be taken that the figure 2·65 ft.-lbs. per second per pound of bird, is here demonstrated to be that which a real bird, with 2 square feet of wing per lb. of bird, flying at 60 feet per second, would or must exert. All that is professed to be shown is, that the bird may be able to fly with the exertion of two or three ft.-lbs. per second per pound, and that 30 to 1300 are quite unnecessary. Further, it appears that, within a very large range of the values of $p\mathrm{S}\cos\theta$, that is frequency of flapping, stroke, and lag of phase of stroke relatively to phase of bird's body motion and twisting of wing, there may be little comparative variation in the horse-power —*e.g.*, between frequency per minute 500 and frequency 250 (supposing $\mathrm{S}\cos\theta = 1$), there would only be a change of work per second from 2·51 to 2·55—scarcely 2 per cent.

This seems to bear on the explanation of the great range of variation in the manner of birds' use of their wings; some (as pigeons and ducks) flapping very fast, others (as seagulls and herons) flapping but slowly, without any physiological reason for expecting material differences in the rate at which their muscles can give out energy.

Fig. 2.



Abscissæ W Foot pounds per pound of Load carried.

Ordinates {AB and CD are $pS\cos\theta$. EF and GH " μ

There are, no doubt, plenty of other reasons, but until experiments can be made on planes or other surfaces flapped by pivoting on an axis, numerical data for conditions of calculation agreeing more nearly than those here used with conditions of fact, seem to be wanting.

Another point brought out is that $pS\cos\theta$ has a minimum value as well as W, but μ has not. This means that the phase of the wing motion, relatively to the twisting of the wing (on which the angle α depends) is important, and if these are out of beat, so to speak, by more than a certain amount, with a given frequency and stroke, the flight cannot be maintained at all. The values of μ which exceed

unity imply (on reference to the equation $\alpha = m\,(1 - \mu \cos pt)$) that there is a reversed pressure on the wing during part of the stroke, as shown in fig. 1 at the third position, where P_3 is directed partly downwards. This condition is however unfavourable, probably in all cases, to economy of labour, though it may be favourable to forward propulsion. At all economical rates of working μ is quite small, that is, the inclination of wing plane to wing path varies but little.

It may also be observed that, for any value of $p\mathrm{S}\cos\theta$, there are two of W, one of which lies close to its least possible value, and the other is very large—*e.g.*, for $p\mathrm{S}\cos\theta = 15$, $\mathrm{W} = 2{\cdot}65$ or about 40, so that a person who was going on observed values of $p\mathrm{S}$ and took $\cos\theta$ to be nearly unity, might easily overlook the small value of W, and base his estimate of work on the large one. This is the more likely as the curve of $p\mathrm{S}\cos\theta$ is of the third degree, and the solving of a cubic by a direct process, especially when there are three real roots, is troublesome, and avoided in consequence. Whether anything of this kind occurred in Navier's or other earlier investigations, the present writer is unable to say. Langley says that Navier added the work done, here expressed by the right-hand side of equation (A) to that on the other to find the whole. Why it should have seemed necessary to do this is not clear, for if the wing has no sensible mass, as here assumed, the pressure of the engine downwards cannot exceed F, and no work can be done by a vertically moving engine, except that expressed on the right-hand side of equation (A). The way in which forward motion is maintained bears some analogy to the process of so-called "invisible" skating, and to screw propulsion, wherein it would be manifestly absurd to add the work done against the ship's resistance to that given by the turning moment applied to the screw, to find the horse-power of the engine. In fact, the construction of Langley's whirling table virtually made the aëroplane a portion of a screw propeller with axis vertical, and his coefficient k has, bound up in it, the inefficiency of the propeller in converting horse-power of turning couple into thrust.

With regard to the absolute magnitude of the figure $2{\cdot}5$ as the least value possible for W in the case illustrated by fig. 2, it is of importance to note that this depends on m, and m has been made $= \dfrac{1}{2k\mathrm{AV}^2}$ on the strength of the equation

$$\frac{d^2\mathrm{Z}}{dt} = g(\mathrm{P}_\alpha - 1) = g\{2k\mathrm{AV}^2 m(1 - \mu\cos pt) - 1\}$$

which again depends on the assumption that the downward acceleration of the body attached to the aeroplane would be g, irrespective of the horizontal velocity of the plane, if the angle of inclination of the plane to its path were zero. Langley's experiments on the "Plane

Dropper" seem to show that this is by no means the case, but, unfortunately, he was unable, through uncontrollable circumstances, to determine the proper value, which is some function of the velocity, and may possibly be so small as $g/3$ at his highest velocities. If so mV which is the minimum value of W requires corresponding reduction, and, in the case taken, the scale of W is about three times too large, reducing the necessary work to about 1 ft.-lb. per pound carried or thereabouts. It is now evident how the energy expenditure of birds may lie within quite reasonable limits, though we cannot, with any approach to accuracy, make the calculations for the design of a flying machine with flapping wings. We can say it will not in a given case cost a pound an hour for power, but we cannot tell whether it will cost only sixpence, or as much as half-a-crown or three shillings.

To allow for head resistance of the bird's body some term must be added to the left-hand side of equation (A), which will alter the absolute term of equation (C) and the $\frac{1}{2}$ under the square root in equation (E) so as to increase W, and a number of small corrections of other kinds might be made, which, however, considering the roughness of the numerical data we have to go upon, are not worth taking into account.

The justification of the assumptions as to the smallness of a and s is best exhibited by a numerical example. Taking then $\mathrm{AV} = 120$ with $\mathrm{A} = 2$ square feet per pound of bird, and assuming $\mathrm{S} = 0{\cdot}5$ (a stroke of 1 foot) we find that if $p = 30$ (about 300 downward strokes per minute, which sea-gulls in regular flight often exceed) and $\cos\theta = 1$, $\mathrm{S}p\cos\theta = 15$, and $\mathrm{W} = 2{\cdot}65$; while μ is $0{\cdot}35$, and m is $0{\cdot}041$. The maximum value of a is then $0{\cdot}041(1+0{\cdot}35)$ when $pt = \pi$. This is $0{\cdot}0554$, the circular measure of $3^\circ\ 10'$ nearly. The maximum value of s, the slope of wing path, is $p\mathrm{S}/\mathrm{V} = 0{\cdot}25$; the circular measure of $14^\circ\ 20'$; but these do not occur simultaneously.

It will be found that their sum is a maximum when $pt = 0$ and is then $0{\cdot}28$, the circular measure of 16°; thus lying well below the limit of 20° suggested in the earlier part of this paper. The body-path, on integration, and taking $\mathrm{Z} = 0$ and $d\mathrm{Z}/dt = 0$ initially, is $\mathrm{Z} = -\mu g/p^2(1-\cos pt)$ and the amplitude is $0{\cdot}0125$ foot. The body, therefore, pursues a simple harmonic path, whose average level is about $5/32$ inch below the mean level of the wing-path, and whose total rise and fall from crest to hollow is almost $5/16$ inch. The relative motion of wing and body does not sensibly differ from that of the wing alone.

The investigation of the question as to hovering is far less satisfactory. We might assume, for instance, that the centre of pressure of the wings followed a circular or elliptical orbit, the plane of the wings making a small angle with the tangent to the orbit, and being variable as before, so as to maintain, on the whole, an upward pressure. If we make this angle $a = m\sin pt$, and form the equation $d^2\mathrm{Z}/dt^2 = g(\mathrm{F}-1)$

as before, remembering, however, that though α may be assumed small, the angle of slope, s, of the wing path, is not, we shall find, if the orbit be taken to be circular, $m = 1/kV^2$ where V is the velocity of the motion along the orbit. In order that m may have values similar to those in the case of progressive flight, the frequency of flapping must be much higher, twice as fast, or thereabouts. But there are several considerations which render results of this kind much more unsatisfactory than those previously obtained. In the first place the motion assumed is very much less like that of a bird's wing than in the former case, inasmuch as it involves complete revolution of the plane of the wing about a horizontal axis in its own plane, and, in the second place, the edges of the plane are continually cutting across the eddies, created by their previous motion, in a very different way from that in which they interfere with those created in progressive flight, so that the numerical value of k, and the value of the function of V which is put down as 1 in the formula $d^2Z/dt^2 = g(F-1)$ are really unknown, and cannot be inferred from experiments with soaring planes, or even propellers. We can only expect that the actual numbers will be tolerably like those given by Langley's and other such experiments.

All that can be inferred, therefore, is that, provided a sufficiently high speed of flapping is attainable, we may reasonably anticipate that the horse-power for hovering need not differ very materially from that for progressive flight. Internal stresses in the wings or machinery, due to their inertia, as well as physiological difficulties connected with high velocities of reciprocation, may also come in to limit the rate of flapping, and prevent an equally economical rate of working being attainable in hovering, as in progressive flight. A considerable simplification of the mere mathematics could have been effected by at once assuming that the arrangement for altering the angle α was so contrived as to make the pressure P_α, or its vertical component, F, an assigned function of the time; but this has the objection that, there being then no expressly formulated relation between α and P or F, experimental evidence would be wanting to settle the values of the numerical constants involved. In fact, the more the mathematical work is generalized, the less definite do the numerical results become, in the present state of our experimental knowledge, and this must form the writer's apology for working out, in a mathematically clumsy way, so limited a case of a problem which has been treated already, in a much more powerful and general way, by far abler hands. For example, Thomson and Tait mention, as an example of motion of a solid in a liquid, the vibratory and irregular movements of such an object as an oyster shell sinking pretty slowly with a swaying motion when thrown flatwise into water. The rate of expenditure of energy necessary to prevent its sinking would, if these motions were forcibly produced,

evidently practically coincide with the rate at which gravity works, when the rate of sinking is, on an average, uniform, and should be quite small. The case of hovering flight of a bird is manifestly closely analogous, and that of progressive flight is similarly closely analogous to that of the same oyster shell, or a piece of slate, projected under water and forcibly maintained in a state of rhythmic motion. The only use of the investigation given in the present paper is to reduce the matter to figures, in a case sufficiently simple to enable us to use numbers supplied by existing experimental results.

[*Note added March* 21, 1899.—The error alluded to in the previous note was detected by checking equation (A) and that for H to see if the horizontal component of the acceleration was exactly periodic. Equation (C), as now given correctly, is the condition in question, with the additional information that each side is the work done. Mr. F. Purser, F.T.C.D., suggested that the wings as well as the body might be supposed heavy. The author has examined this point, and finds that, supposing, as before, that the engine, say a cylinder and piston, to whose rod the aëroplane is attached, works vertically, no difference is made in equation (C). The force P_α being inclined, has, or may have, a moment about the centre of inertia of the whole machine; the effects of this should be wholly periodic, but inasmuch as it appears from experiment and otherwise that P does not pass through the centre of figure of the plane, and moves about in some way depending on the angle α, it would be impossible to test this condition without assuming details of dimensions, &c. Any actual bird or flying machine, must have some steering apparatus, capable of correcting disturbances of a rotational kind in both the vertical and the horizontal plane, but its management plainly cannot involve any material alteration in the work to be done. The author conceives it to be possible, if not highly probable, that the motion in flying is, in respect of direction, unstable. Small periodic terms may be assumed also to be added to V, but lead to nothing up to the order of quantities involved in neglecting the difference between cosine and radius.]